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Secondary flow in a curved tube

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The work of Dean and that of McConalogue & Srivastava on the steady motion of an incompressible fluid through a curved tube of circular cross-section is extended through the entire range of Reynolds numbers for which the flow is laminar. The coupled nonlinear system of partial differential equations which defines the motion is solved numerically by finite differences. Computer calculations are described and physical implications are discussed.

1. Introduction

The flow of a fluid in a curved tube has been of broad interest both experimentally (see, e.g. Eustice 1911; Taylor 1929) and theoretically (see, e.g. Dean 1927, 1928; McConalogue 1970; McConalogue & Srivastava 1968). In this paper we shall study, in particular, the steady secondary flow of an incompressible fluid through a pipe of circular cross-section which is coiled in a circle. Our approach will be numerical and will be applied to the particular model studied qualitatively by Dean (1927) and numerically by McConalogue & Srivastava (1968). The method to be used will be a finite-difference technique (Greenspan 1968, 1969) and will be both simpler and more comprehensive than that of McConalogue & Srivastava.

Mathematically, the problem to be considered is formulated as follows. Consider a pipe of circular cross-section, coiled in the form of a circle. As shown in figure 1, let the axis of the circle in which the pipe is coiled be OY and let C be the centre of the section of the pipe formed by a plane through OY which makes an angle θ with a fixed axial plane. Let OC be of length L, and let the radius of the cross-section be a. The co-ordinates of any point P of the cross-section are denoted by orthogonal co-ordinates (r', α, θ), where r' is the distance CP and α is the angle CP makes with OC. Let the velocity components at P be (U, V, W), where U is in the direction CP, V is perpendicular to U and in the plane of the cross-section, and W is perpendicular to this plane. The motion of the fluid is assumed to be due to a fall in pressure in the direction of increasing θ. It is assumed also that a/L is relatively small (McConalogue & Srivastava 1968); that U, V and W are independent of θ; and that the motion is steady. Setting

$$r'U = \partial f/\partial \alpha, \quad V = -\partial f/\partial r', \tag{1.1}$$

where f, the stream function of the secondary flow, is a function only of r' and α; defining the constant D by

$$D = 4R(2a/L)^{\frac{1}{2}}, \tag{1.2}$$

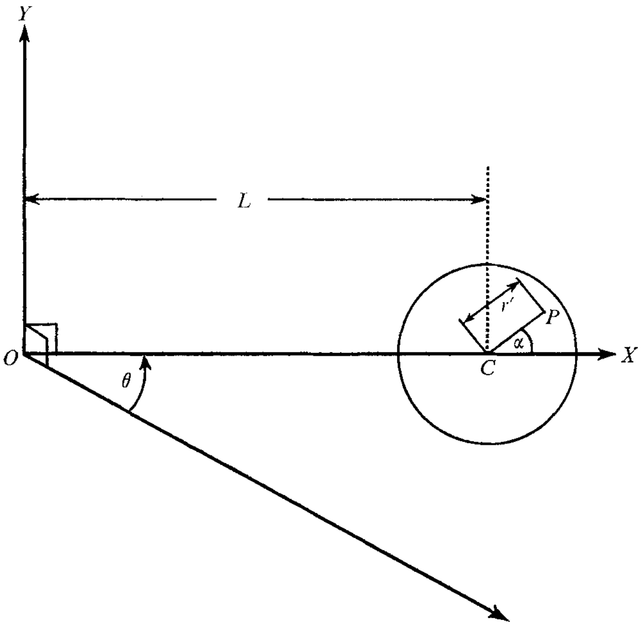

FIGURE 1. Co-ordinate system

where R is a given Reynolds number; and introducing the non-dimensionsional variables

$$f = \nu\phi, \quad W = w(\nu^2 L/2a^3)^{\frac{1}{2}}, \quad r' = ar, \tag{1.3}$$

where ν is the kinematic viscosity, yields the following equations of motion (McConalogue & Srivastava 1968):

$$\nabla_1^2 w + D = \frac{1}{r}\left(\frac{\partial\phi}{\partial\alpha}\frac{\partial w}{\partial r} - \frac{\partial\phi}{\partial r}\frac{\partial w}{\partial\alpha}\right), \tag{1.4}$$

$$-\nabla_1^4\phi = \frac{1}{r}\left(\frac{\partial\phi}{\partial r}\frac{\partial}{\partial\alpha} - \frac{\partial\phi}{\partial\alpha}\frac{\partial}{\partial r}\right)\nabla_1^2\phi + w\left(\frac{\partial w}{\partial r}\sin\alpha + \frac{\partial w}{\partial\alpha}\frac{\cos\alpha}{r}\right), \tag{1.5}$$

in which

$$\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\alpha^2}. \tag{1.6}$$

The boundary constraints at $r = 1$ are

$$w = \phi = \partial\phi/\partial r = 0. \tag{1.7}$$

The problem, then, is to solve the coupled nonlinear partial differential equations (1.4) and (1.5) subject to boundary conditions (1.7).

Physically, the experiments of Eustice (1911) and Taylor (1929) have shown that, for curved tubes, flow can be laminar for much greater Reynolds numbers than is the case of a straight tube, and since Taylor (1929) showed that the critical Reynolds number rose to about 5000 for the case $L/a = 31{\cdot}9$, interest has centred on the following range of D:

$$0 \leqslant D \leqslant 5000. \tag{1.8}$$

Thus far, convergent results have been obtained only by Dean (1927) for $0 \leqslant D \leqslant 96$ and by McConalogue & Srivastava (1968) for $96 \leqslant D \leqslant 605{\cdot}72$.

In our development of a numerical method which will be convergent for the entire range (1·8), we shall be motivated by the powerful difference methods and supporting theory which exist for second-order elliptic equations (Greenspan 1968). For this reason, let us rewrite (1.4) and (1.5) as the following system of second-order equations:

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\alpha^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}=-\Omega, \tag{1.9}$$

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\alpha^2}+\frac{1}{r}\left[\left(\frac{\partial\phi}{\partial r}\frac{\partial w}{\partial\alpha}\right)+\left(1-\frac{\partial\phi}{\partial\alpha}\right)\frac{\partial w}{\partial r}\right]=-D, \tag{1.10}$$

$$\frac{\partial^2\Omega}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\Omega}{\partial\alpha^2}+\frac{1}{r}\left[\frac{\partial\phi}{\partial r}\frac{\partial\Omega}{\partial\alpha}+\left(1-\frac{\partial\phi}{\partial\alpha}\right)\frac{\partial\Omega}{\partial r}\right]=w\left(\sin\alpha\frac{\partial w}{\partial r}+\frac{\cos\alpha}{r}\frac{\partial w}{\partial\alpha}\right). \tag{1.11}$$

Observe that (1.9)–(1.11) are, in fact, valid only for $r > 0$. The singularity at $r = 0$ is, nevertheless, not physical but geometric, and is due to recasting of the equations

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=-\Omega, \tag{1.9a}$$

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\left(\frac{\partial\phi}{\partial x}\frac{\partial w}{\partial y}-\frac{\partial\phi}{\partial y}\frac{\partial w}{\partial x}\right)=-D, \tag{1.10a}$$

$$\frac{\partial^2\Omega}{\partial x^2}+\frac{\partial^2\Omega}{\partial y^2}+\left(\frac{\partial\phi}{\partial x}\frac{\partial\Omega}{\partial y}-\frac{\partial\phi}{\partial y}\frac{\partial\Omega}{\partial x}\right)=w\frac{\partial w}{\partial y} \tag{1.11a}$$

into polar co-ordinates.

However (1.9*a*)–(1.11*a*) yield, readily, the symmetry relationships

$$\phi(x,y)=-\phi(x,-y), \tag{1.12}$$

$$\Omega(x,y)=-\Omega(x,-y), \tag{1.13}$$

$$w(x,y)=w(x,-y), \tag{1.14}$$

which, in turn, will allow us to study our problem on the semicircle defined by $0\leqslant r\leqslant 1$, $0\leqslant\alpha\leqslant\pi$. Indeed, from (1.12) and (1.13), one has immediately, in rectangular co-ordinates, that

$$\phi(x,0)=\Omega(x,0)=0. \tag{1.15}$$

2. Difference-equation approximations

Fundamental to the method to be developed is the approximation of differential equations (1.9)–(1.11) and (1.10*a*) by difference equations which are associated with diagonally dominant, linear algebraic systems. This will be accomplished by using a combination of central-, forward- and backward-difference approximations for derivatives as follows, in the same spirit as in Greenspan (1969).

Consider first $r = 0$ and (1.10*a*). In rectangular co-ordinates, and for $\Delta r > 0$, let the five points $(0,0)$, $(\Delta r,0)$, $(0,\Delta r)$, $(-\Delta r,0)$ and $(0,-\Delta r)$ be numbered 0, 1, 2, 3 and 4, respectively. Then, in the usual subscript notation (Greenspan 1969) approximate the second-order derivative terms at $(0,0)$ by

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\frac{-4w_0+w_1+w_2+w_3+w_4}{(\Delta r)^2}. \tag{2.1}$$

Next, set
$$\frac{\partial\phi}{\partial x} = \frac{\phi_1-\phi_3}{2\Delta r}, \quad \frac{\partial\phi}{\partial y} = \frac{\phi_2-\phi_4}{2\Delta r} \tag{2.2}$$
and
$$\epsilon = \phi_1-\phi_3, \tag{2.3}$$
$$\beta = \phi_2-\phi_4. \tag{2.4}$$
Then, approximate $\partial w/\partial y$ and $\partial w/\partial x$ by
$$\frac{\partial w}{\partial y} = \begin{cases}(w_2-w_0)/\Delta r & (\epsilon \geqslant 0),\\ (w_0-w_4)/\Delta r & (\epsilon < 0),\end{cases} \tag{2.5}$$
$$\frac{\partial w}{\partial x} = \begin{cases}(w_0-w_3)/\Delta r & (\beta \geqslant 0),\\ (w_1-w_0)/\Delta r & (\beta < 0).\end{cases} \tag{2.6}$$
If one now defines the quantities A, B and C by
$$A = -4-\tfrac{1}{2}|\epsilon|-\tfrac{1}{2}|\beta|, \tag{2.7}$$
$$B = 1+\tfrac{1}{2}|\epsilon|, \tag{2.8}$$
$$C = 1+\tfrac{1}{2}|\beta|, \tag{2.9}$$
then the difference approximation of (1.10a) which results is
$$\left.\begin{aligned} Aw_0+w_1+Bw_2+Cw_3+w_4 &= -(\Delta r)^2 D \quad (\epsilon \geqslant 0, \quad \beta \geqslant 0),\\ Aw_0+Cw_1+Bw_2+w_3+w_4 &= -(\Delta r)^2 D \quad (\epsilon \geqslant 0, \quad \beta < 0),\\ Aw_0+w_1+w_2+Cw_3+Bw_4 &= -(\Delta r)^2 D \quad (\epsilon < 0, \quad \beta \geqslant 0),\\ Aw_0+Cw_1+w_2+w_3+Bw_4 &= -(\Delta r)^2 D \quad (\epsilon < 0, \quad \beta < 0).\end{aligned}\right\} \tag{2.10}$$

Consider, next, $r > 0$ and (1.9)–(1.11). For given positive values of Δr and $\Delta\alpha$, let the five polar points (r, α), $(r+\Delta r, \alpha)$, $(r, \alpha+\Delta\alpha)$, $(r-\Delta r, \alpha)$ and $(r, \alpha-\Delta\alpha)$ be numbered $0, 1, 2, 3$ and 4, respectively. Let the second-order derivatives in (1.9)–(1.11) be approximated by
$$\left.\frac{\partial^2\phi}{\partial r^2}\right|_0 = \frac{\phi_1-2\phi_0+\phi_3}{(\Delta r)^2}, \quad \left.\frac{\partial^2\phi}{\partial \alpha^2}\right|_0 = \frac{\phi_2-2\phi_0+\phi_4}{(\Delta\alpha)^2}, \tag{2.11}$$
$$\left.\frac{\partial^2 w}{\partial r^2}\right|_0 = \frac{w_1-2w_0+w_3}{(\Delta r)^2}, \quad \left.\frac{\partial^2 w}{\partial \alpha^2}\right|_0 = \frac{w_2-2w_0+w_4}{(\Delta\alpha)^2}, \tag{2.12}$$
$$\left.\frac{\partial^2\Omega}{\partial r^2}\right|_0 = \frac{\Omega_1-2\Omega_0+\Omega_3}{(\Delta r)^2}, \quad \left.\frac{\partial^2\Omega}{\partial \alpha^2}\right|_0 = \frac{\Omega_2-2\Omega_0+\Omega_4}{(\Delta\alpha)^2}. \tag{2.13}$$
In (1.9), set
$$\left.\frac{\partial\phi}{\partial r}\right|_0 = \frac{\phi_1-\phi_0}{\Delta r}. \tag{2.14}$$
Then, in (1.10), use
$$\left.\left(1-\frac{\partial\phi}{\partial\alpha}\right)\right|_0 = \frac{2\Delta\alpha-\phi_2+\phi_4}{2\Delta\alpha}, \quad \left.\left(\frac{\partial\phi}{\partial r}\right)\right|_0 = \frac{\phi_1-\phi_3}{2\Delta r}. \tag{2.15}$$
Now, define γ and δ by
$$\phi_1-\phi_3 = \gamma, \quad 2\Delta\alpha-\phi_2+\phi_4 = \delta \tag{2.16}$$
and approximate $\partial w/\partial\alpha$ and $\partial w/\partial r$ in (1.10) as follows:
$$\frac{\partial w}{\partial \alpha} = \left\{\begin{matrix}(w_2-w_0)/\Delta\alpha & (\gamma \geqslant 0),\\ (w_0-w_4)/\Delta\alpha & (\gamma < 0),\end{matrix}\right\} \tag{2.17}$$
$$\frac{\partial w}{\partial r} = \left\{\begin{matrix}(w_1-w_0)/\Delta r & (\delta \geqslant 0),\\ (w_0-w_3)/\Delta r & (\delta < 0).\end{matrix}\right\} \tag{2.18}$$

For (1.11), use (2.13), (2.14), (2.15) and, with w replaced by Ω, (2.17) and (2.18). Finally, in (1.11), approximate $\partial w/\partial r$ and $\partial w/\partial \alpha$ by

$$\frac{\partial w}{\partial r} = \frac{w_1 - w_3}{2\Delta r}, \quad \frac{\partial w}{\partial \alpha} = \frac{w_2 - w_4}{2\Delta \alpha}. \tag{2.19}$$

If one defines the quantities E, F, G, H, I and J by

$$E = -\frac{2}{(\Delta r)^2} - \frac{2}{r^2(\Delta\alpha)^2} - \frac{|\gamma| + |\delta|}{2r\,\Delta r\,\Delta\alpha}, \quad F = \frac{1}{(\Delta r)^2} + \frac{|\delta|}{2r\,\Delta r\,\Delta\alpha}, \quad H = \frac{1}{(\Delta r)^2},$$

$$G = \frac{1}{r^2(\Delta\alpha)^2} + \frac{|\gamma|}{2r\,\Delta r\,\Delta\alpha}, \quad I = \frac{1}{r^2(\Delta\alpha)^2}, \quad J = w_0 \sin\alpha\left(\frac{w_1 - w_3}{2\Delta r}\right) + \frac{w_0\cos\alpha}{r}\left(\frac{w_2 - w_4}{2\Delta\alpha}\right),$$

then the respective difference approximations of (1.9)–(1.11) which result are

$$\left[-\frac{2}{(\Delta r)^2} - \frac{2}{r^2(\Delta\alpha)^2} - \frac{1}{r\Delta r}\right]\phi_0 + \left[\frac{1}{(\Delta r)^2} + \frac{1}{r\Delta r}\right]\phi_1 + \frac{1}{r^2(\Delta\alpha)^2}\phi_2 + \frac{1}{(\Delta r)^2}\phi_3 + \frac{1}{r^2(\Delta\alpha)^2}\phi_4 = -\Omega_0, \tag{2.20}$$

$$\left.\begin{aligned}
Ew_0 + Fw_1 + Gw_2 + Hw_3 + Iw_4 &= -D \quad (\gamma \geqslant 0, \quad \delta \geqslant 0),\\
Ew_0 + Hw_1 + Gw_2 + Fw_3 + Iw_4 &= -D \quad (\gamma \geqslant 0, \quad \delta < 0),\\
Ew_0 + Fw_1 + Iw_2 + Hw_3 + Gw_4 &= -D \quad (\gamma < 0, \quad \delta \geqslant 0),\\
Ew_0 + Hw_1 + Iw_2 + Fw_3 + Gw_4 &= -D \quad (\gamma < 0, \quad \delta < 0).
\end{aligned}\right\} \tag{2.21}$$

$$\left.\begin{aligned}
E\Omega_0 + F\Omega_1 + G\Omega_2 + H\Omega_3 + I\Omega_4 &= J \quad (\gamma \geqslant 0, \quad \delta \geqslant 0),\\
E\Omega_0 + H\Omega_1 + G\Omega_2 + F\Omega_3 + I\Omega_4 &= J \quad (\gamma \geqslant 0, \quad \delta < 0),\\
E\Omega_0 + F\Omega_1 + I\Omega_2 + H\Omega_3 + G\Omega_4 &= J \quad (\gamma < 0, \quad \delta \geqslant 0),\\
E\Omega_0 + H\Omega_1 + I\Omega_2 + F\Omega_3 + G\Omega_4 &= J \quad (\gamma < 0, \quad \delta < 0).
\end{aligned}\right\} \tag{2.22}$$

3. The numerical method

As shown in figure 2, let R be the semicircular region defined by

$$0 < r < 1, \quad 0 < \alpha < \pi$$

and let S be the boundary of R. For finite positive grid sizes Δr and $\Delta\alpha$, where $(\Delta r)^{-1}$ and $\frac{1}{2}\pi(\Delta\alpha)^{-1}$ are integers, construct and number in the usual way the interior polar grid points R_h and the boundary polar grid points S_h.

In general, we shall construct on $R_h \cup S_h$ a triple sequence of discrete functions

$$\phi^{(0)}, \phi^{(1)}, \phi^{(2)}, \ldots, \tag{3.1}$$

$$w^{(0)}, w^{(1)}, w^{(2)}, \ldots, \tag{3.2}$$

$$\Omega^{(0)}, \Omega^{(1)}, \Omega^{(2)}, \ldots, \tag{3.3}$$

with the property that, for some integer k, and for given positive tolerances ϵ_1, ϵ_2 and ϵ_3,

$$|\phi^{(k)} - \phi^{(k+1)}| < \epsilon_1, \tag{3.4}$$

$$|w^{(k)} - w^{(k+1)}| < \epsilon_2, \tag{3.5}$$

$$|\Omega^{(k)} - \Omega^{(k+1)}| < \epsilon_3 \tag{3.6}$$

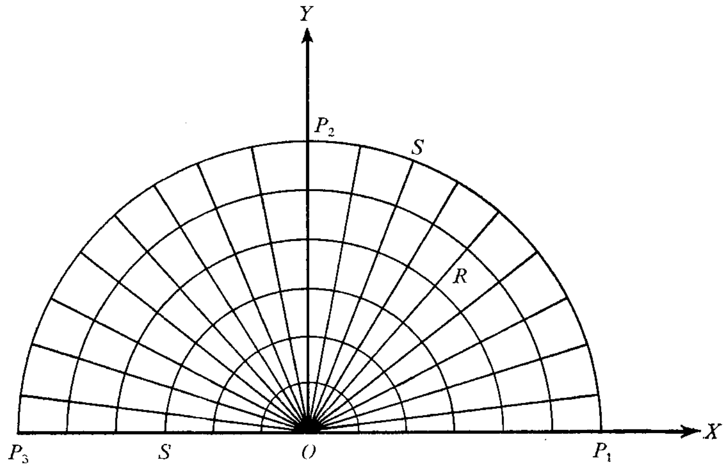

FIGURE 2. Grid system.

uniformly on $R_h \cup S_h$. Each of the discrete functions in sequences (3.1)–(3.3) will be called an outer iterate. For $j = 1, 2, \ldots$, each $\phi^{(j)}$ will be a solution of (2.20), each $w^{(j)}$ will be a solution of (2.10) or (2.21), and each $\Omega^{(j)}$ will be a solution of (2.22). Numerical convergence to the tolerances (3.4)–(3.6) will yield the discrete approximate solutions $\phi^{(k+1)}$, $w^{(k+1)}$ and $\Omega^{(k+1)}$ for ϕ, w and Ω, respectively.

Specifically, the algorithm proceeds in the following fashion, with the origin being expressed in rectangular co-ordinates and all other points being expressed in polar co-ordinates.

Step 1. Define $\phi^{(0)}$, $w^{(0)}$ and $\Omega^{(0)}$ arbitrarily on $R_h \cup S_h$ except that $\phi^{(0)} = 0$ on S_h, $w^{(0)} = 0$ at each point of S_h for which $r = 1$, and $\Omega^{(0)} = 0$ at each point of S_h which is also a point of the X axis.

Step 2. At each point of S_h, set

$$\phi = 0. \tag{3.7}$$

At each point of R_h for which $r = 1-\Delta r$, set

$$\phi(1-\Delta r, \alpha) = \tfrac{1}{4}\phi(1-2\Delta r, \alpha). \tag{3.8}$$

On the remaining points of R_h, write down (2.20) with Ω_0 replaced by $\Omega_0^{(k)}$. Solve the linear algebraic system so generated by *SOR* (Greenspan 1968) with over-relaxation factor r_ϕ and denote the solution by $\bar{\phi}^{(k+1)}$. Then, define $\phi^{(k+1)}$ on $R_h \cup S_h$ by the smoothing formula

$$\phi^{(k+1)} = \xi_1 \phi^{(k)} + (1-\xi_1)\bar{\phi}^{(k+1)} \quad (0 \leqslant \xi_1 \leqslant 1). \tag{3.9}$$

Step 3. At each point of S_h for which $r = 1$, set $w = 0$. At the origin write down (2.10) with each ϕ_i replaced by the known value $\phi_i^{(k+1)}$ given by (3.9), with ϕ_4 replaced by $-\phi_2$, and with w_4 replaced by w_2. On the remaining points of R_h, write down (2.21) with ϕ_i replaced by $\phi_i^{(k+1)}$. On the remaining points of S_h, write down (2.21) with ϕ_i replaced by $\phi_i^{(k+1)}$, with ϕ_4 replaced by $-\phi_2$ and w_4 replaced by w_2 between O and P_1, and with ϕ_2 replaced by $-\phi_4$ and w_2 replaced by w_4 between O and P_3.

D	ϵ_1	ϵ_2	ϵ_3	r_ϕ	r_w	r_Ω	ξ_1	ξ_2	ξ_3	ξ_4	Number outer iterations for convergence
10	10^{-5}	10^{-3}	10^{-4}	1·5	1·8	1·5	0·1	0·1	0·1	0·1	8
100	2×10^{-4}	5×10^{-3}	5×10^{-3}	1·5	1·7	1·5	0·1	0·1	0·1	0·1	8
250	2×10^{-3}	2×10^{-2}	4×10^{-2}	1·5	1·8 on iteration 1 only, then 1·5	1·5	0·1	0·1	0·1	0·1	9
500	4×10^{-3}	4×10^{-2}	8×10^{-2}	1·5	1·5	1·5	0·1	0·1	0·1	0·1	13
1000	5×10^{-3}	5×10^{-2}	17×10^{-2}	1·5	1·5	1·5	0·5	0·1	0·1	0·5	21
2000	7×10^{-3}	8×10^{-2}	3×10^{-1}	1·5	1·5	1·3	0·5	0·1	0·1	0·5	25
5000	10^{-2}	15×10^{-2}	6×10^{-1}	1·5	1·5	1·3	0·3	0·1	0·1	0·7	25

TABLE 1

Solve the linear algebraic system generated above by *SOR* using r_w as over-relaxation factor, and denote the solution by $\overline{w}^{(k+1)}$. Then, define $w^{(k+1)}$ on $R_h \cup S_h$ by

$$w^{(k+1)} = \xi_2 w^{(k)} + (1-\xi_2)\,\overline{w}^{(k+1)} \quad (0 \leqslant \xi_2 \leqslant 1). \tag{3.10}$$

Step 4. At each point of S_h for which $r = 1$, set

$$\overline{\Omega}^{(k+1)}(1,\alpha) = -2(\Delta r)^{-2}\phi^{(k+1)}(1-\Delta r,\alpha).$$

Then define $\Omega^{(k+1)}$ on this set of points by

$$\Omega^{(k+1)} = \xi_3\,\Omega^{(k)} + (1-\xi_3)\,\overline{\Omega}^{(k+1)}, \quad 0 \leqslant \xi_3 \leqslant 1. \tag{3.11}$$

Step 5. At the points of S_h not considered in step 4, which are all on the X axis, set $\Omega = 0$. At each point of R_h, write down (2.22) with ϕ_i replaced by $\phi_i^{(k+1)}$, with w_i replaced by $w_i^{(k+1)}$, and with Ω at each boundary point for which $r = 1$ determined by (3.11). Solve the linear algebraic system so generated by *SOR* with over-relaxation factor r_Ω. Denote the solution by $\overline{\Omega}^{(k+1)}$. Finally, define $\Omega^{(k+1)}$ on the set of points not included in step 4 by

$$\Omega^{(k+1)} = \xi_3\,\Omega^{(k)} + (1-\xi_4)\,\overline{\Omega}^{(k+1)} \quad (0 \leqslant \xi_4 \leqslant 1). \tag{3.12}$$

Step 6. Do steps 2–5 for $k = 0, 1, 2, \ldots$. Terminate when (3.4)–(3.6) are satisfied.

For a complete FORTRAN program of the above algorithm, see Schubert (1972).

4. Examples and results

A large variety of examples using the method of §3 were run on the UNIVAC 1108 at the University of Wisconsin and a selection of convergent ones in which $D = 10$, 100, 250, 500, 1000, 2000 and 5000 are summarized in table 1. Economic

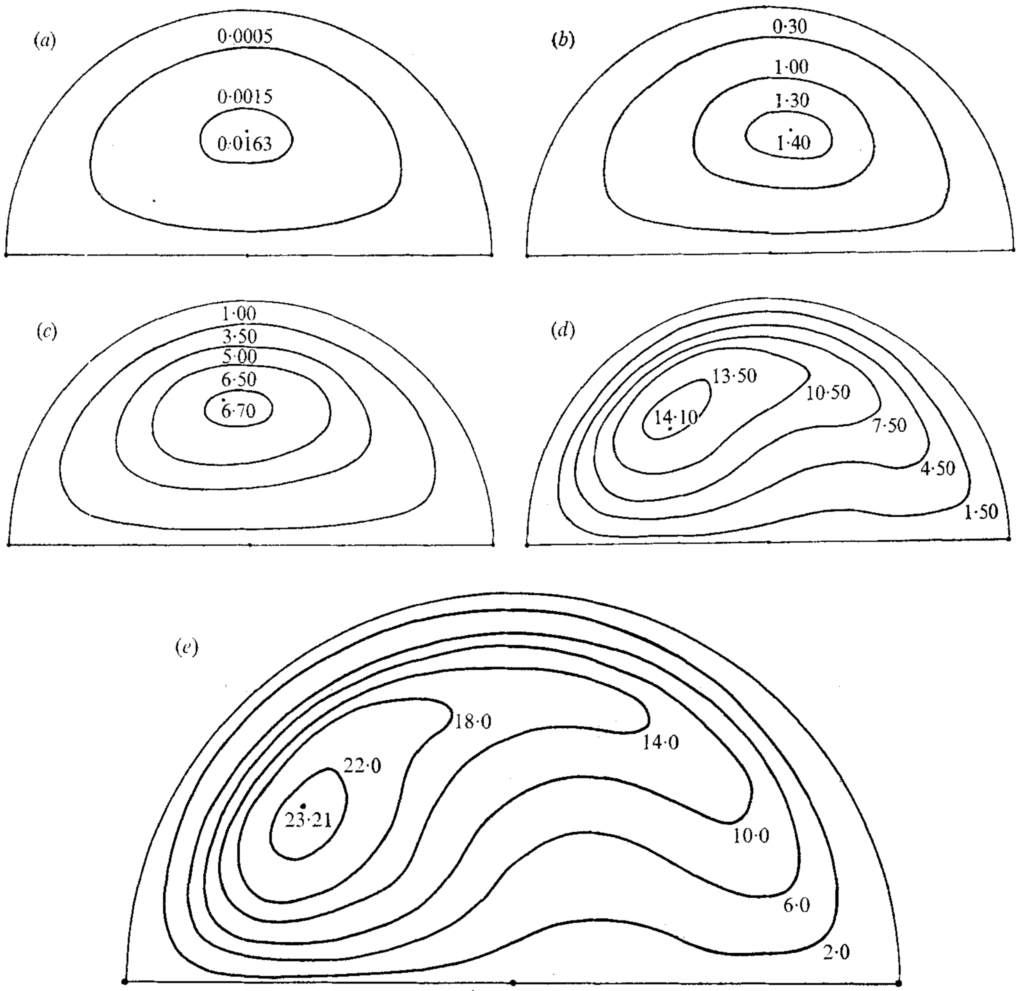

FIGURE 3. Constant-ϕ curves for (a) $D = 10$, (b) $D = 100$, (c) $D = 500$, (d) $D = 2000$, (e) $D = 5000$.

constraints restricted the grid sizes in each case to $\Delta r = 0{\cdot}1$ and $\Delta\alpha = \frac{1}{18}\pi$, and no case required more than three minutes of computing time. The input values for $D = 10$ were $\phi^{(0)} = w^{(0)} = \Omega^{(0)} = 0$. The input values for any other D in table 1 were the converged results obtained for the previous value of D. The *SOR* tolerances associated with ϕ, w and Ω were set at $\frac{1}{20}\epsilon_i$, $i = 1, 2, 3$, respectively. Graphs of constant-ϕ and constant-w curves for $D = 10$, 100, 500, 2000 and 5000 are given in figures 3 and 4.

Most of the qualitative physical trends observed by McConalogue & Srivastava continue to develop so that, with increasing D, the axial-momentum peak moves well away from the origin, the secondary-flow velocity becomes more uniform in a large central region and there is a considerable reduction in the flux in the curved tube compared with that of the straight tube. The unexpected result is that the core of the constant-ϕ curves exhibits a clockwise motion about the

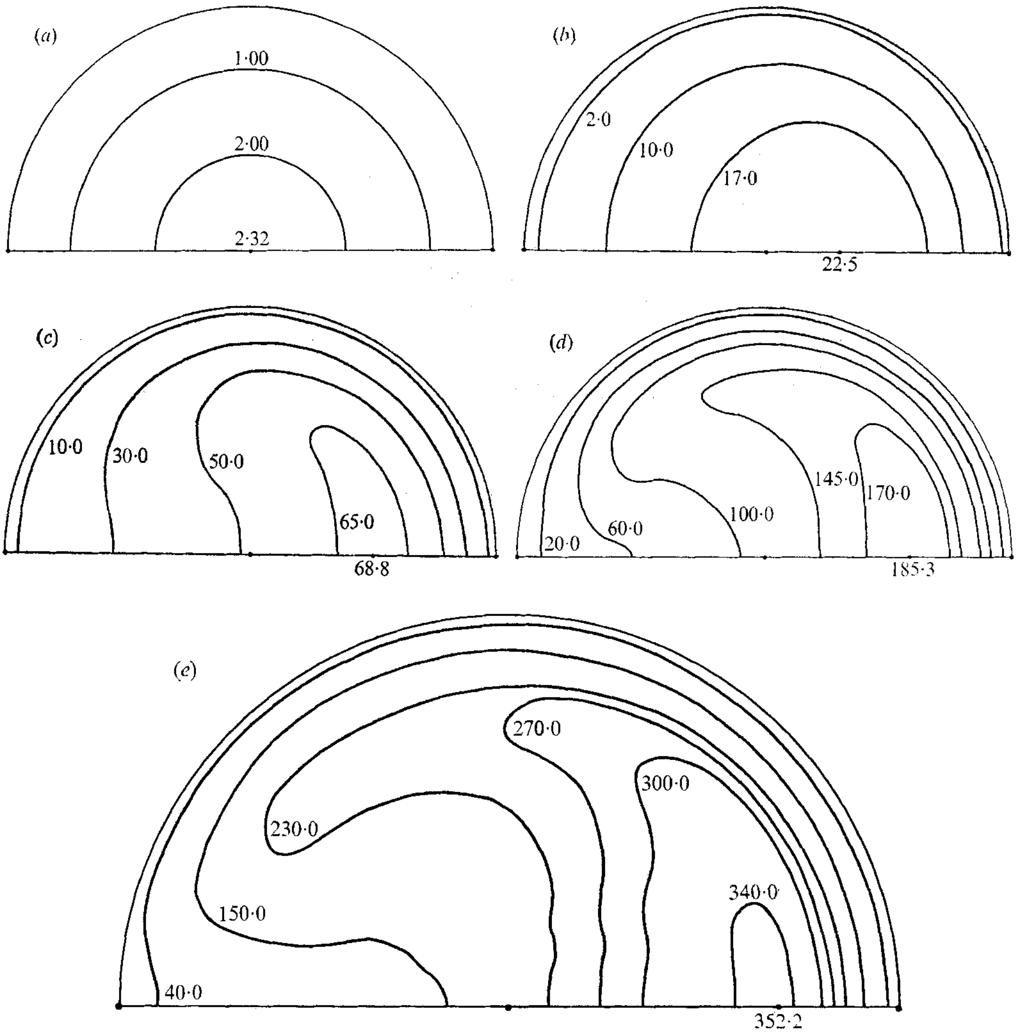

FIGURE 4. Constant-w curves for (a) $D = 10$, (b) $D = 100$, (c) $D = 500$, (d) $D = 2000$, (e) $D = 5000$.

origin up to $D = 500$ and then, for $D \geqslant 500$, reverses to one which is counterclockwise. It is also of interest to note that, for $D = 5000$, the constant-w curves have developed several oscillatory portions near the origin, which would seem to presage the onset of turbulence.

Quantitatively, almost nothing of a precise nature can be said in comparing the results of this paper and those of McConalogue & Srivastava. Of course one can note, for example, that our result $w_{\max} = 22{\cdot}5$ for $D = 100$ corresponds well with their result $w_{\max} = 23{\cdot}4$ for $D = 96$, or that our result $\phi_{\max} = 6{\cdot}7$ for $D = 500$ is comparable to their result $\phi_{\max} = 6{\cdot}81$ for $D = 605{\cdot}72$. Unfortunately, their calculations and ours are for different sets of D values and linear interpolation *cannot* be applied to deduce valid comparisons.

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